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CCT 605: Math Thinking Skills
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Lesson Plan: Functions, Number, Countable and Uncountable Infinities

Learning Objectives:
After this lesson, the student should be able to:

(1) Give the definition of a function and its associated parts (*domain, range, image*).
(2) Explain the differences among *injective* (one-to-one), *surjective* (onto), and *bijective* (one-to-one and onto) functions and express them pictorially.
(3) Give an intuitive definition of number and equinumerosity.
(4) Prove relationships of size for sets of finite cardinality.
(5) Prove basic facts about the cardinality of the natural numbers.
(6) Prove that there are more real numbers than counting numbers.

Materials:
A blackboard and chalk (or equivalent).

1. Lesson Summary

Students learn to tell if two sets of objects are of equal size by constructing a matching rule from one set to the other. The rule they construct is a function. Students will explore the three cases that can arise (which correspond to "is greater than," "is less than," and "is equal to") graphically. Through several guiding examples, students will develop the technology necessary to prove simple statements about the natural numbers—*e.g.*, $0 \leqslant n$ for any natural number n.

By way of analogy, students will be led to consider sets of infinite cardinality. Along the way, students will be able to show both symbolically and geometrically that any two line segments have the same number of points, and that an infinite line has as many points as a finite line segment.

Eventually, students should be able to demonstrate that there are more real numbers than natural numbers using Cantor's diagonalization argument. The lesson leaves many questions open. For instance, are there more fractions or natural numbers? The teacher should cook up her own examples. This guide only provides a skeleton outline. The lessons last for several days, as there is too much material to cram into one.

2. Lesson Guide

Motivating Problem.
The teacher poses the following problem:

> My uncle, who is a rancher, needs me to feed one carrot to each of his llamas. Now the thing is, I don't know how to count, and I'm not sure if I have enough carrots to feed each of the llamas. How can I know if I have more carrots or llamas without counting?

Aim: To motivate the definition of a function.
Goal: To feed (associate) one carrot to one llama.
Question: Could we have fed the same carrot to more than one llama?

Aim: To motivate the definition of injectivity (one-to-oneness).
Question: Could we have fed one llama more than one carrot? Would we be able to tell of which we had more that way?

Aim: To establish a concept of size of a set.
Goal: To look for left overs after an injective pairing.
Question: After I am done feeding each carrot to a distinct llama, how will I know if I had more carrots or llamas? What if there are exactly the same number of carrots and llamas?

Aim: To motivate the definition of surjectivity (onto-ness).
Question: Can someone draw me a picture for the situation when there are more carrots than there are llamas? What do we notice? Can the rule be one-to-one? Could we have a situation where every llama is feed exactly one carrot? What would that mean about the number of carrots and the number of llamas?

Equinumerosity.
By this time, we should be able to move beyond the concrete carrots and llamas example and move to sets of any (more convenient) elements—points, numbers, pictures, etc.

Aim: To motivate the definition of bijectivity.
Goal: There is a function that is one-to-one and onto if and only if two sets are of the same size.
Question: So, how can we prove that two sets are the same size? Think about the carrots and llamas. What will happen if we have exactly the same number of carrots as llamas?

Aim: To establish the relationship greater (less) than or equal to.
Question: How can we show that one collection of things is greater than or equal to another group of things?
Examples: How can we prove that any natural number is greater than or equal to zero? How can we show that any number is equal to itself? Is there more than one way to do it?

Aim: To proof the reflexive, symmetric, and transitive properties of equality.
Question: If there is a bijection between sets X and Y and another between Y and Z, what can you say about the sizes of X compared to itself, compared to Y, compared to Z? What can you say about the size of Y compared to Z; Z compared to Y?

Aim: To count the number of functions from a set into itself.
Question: How many functions are there from one set to itself? How many bijections are there?

Sets of Infinite Cardinality.
Now that we have a method to determine cardinality of a set, we can begin to ask questions about sets with an infinite number of elements.

Aim: To introduce the concept of infinity.
Question: How many natural/counting numbers are there? How do we know that thre really isn't a finite number of them?

Aim: To motivate the definition of countably infinite.
Question: Are there more even numbers or counting numbers? Odd numbers? Perfect squares? Integers? *(Optional)* Primes?

Aim: To extend addition and subtraction to handle the cardinality of the natural numbers ω.
Question: Which set is bigger, the natural numbers or the even numbers without the number 2; the even numbers and the number 1? What should the answers to a statement like $\omega + 1$ or $\omega - 1$ be?

Bijections Again.
Now we are in a position to discuss other sets of (not necessarily countable) infinite cardinality.

Aim: To list other sets of infinite cardinality.
Goal: To talk about finite line segments comprised of an infinite number of points.
Question: Can anyone think of an example from geometry that shows an infinite number of things?

Aim: To show that any two finite line segments are *isomorphic*; *i.e.*, there is a bijection between them.
Question: Name two line segments for me. Are there more points in the first one or in the second one, are there the same number in both, or is it impossible to know? How could you show this using symbols? How could you show this without using symbols (but by geometric arguments)?

Aim: To show that an open line segment and the entire real line are isomorphic.
Question: Are there more points in an open line segment or in the entire real line, or are there the same number of points in each, or is it impossible to know?

Aim: To show that the set of real numbers and a circle are isomorphic.
Question: How many points are there in the segment $[0, 360)$ [or $[0, 2\pi)$] or in a circle?

Aim: To show that the set of real numbers and an infinite line are isomorphic.

Aim: To show there are more real numbers than natural numbers.
Goal: Cantor's diagonalization argument.
Question: How can we show that there are more real numbers than natural numbers? Could we show that there are the same size? Can we write down all the real numbers?

3. Conclusion and Extension

The line of questioning given above can lead a group of students through several, major results in Cantorian set theory. These results found here are by no means exhaustive. And there are several other paths that lead to the same place. Moreover, the subject is rich enough to allow very personalized investigation.

We never showed that the rational numbers are countable, nor did we show that the irrational numbers are uncountable. The fact that $2\mathbb{N} \cong \mathbb{N}$ motivates the following theorem: *A set X is infinite if and only if there exists a proper subset Y such that $X \cong Y$.* Also, we have not come up with a way to represent negative integers. These are only a few of a host of possible observations; it is left to the instructor to decide in which ways he would like to guide his classroom.